Expander Graphs
Exercise Sheet 2

Question 1. Let G be a graph and A the adjacency matrix of G. For any k show that $(A^k)_{u,v}$ is equal to the number of walks of length k from u to v in G.

Question 2. In this question we consider the lazy random walk on an (n,d)-graph G, whose transition matrix is given by $W = \frac{1}{2}(I + \hat{A})$. Show that the eigenvalues of W satisfy $1 = \lambda_1 > \lambda_2 \ldots \geq \lambda_n \geq 0$.

Let $N = I - \hat{A}$. Show that the eigenvalues of N are $\nu_i = 2(1 - \lambda_i)$. For any initial distribution $\boldsymbol{\pi^0}$, by expanding out over an eigenbasis for N, show that $\boldsymbol{\pi^t} \to \boldsymbol{v^1} = \boldsymbol{u}$.

(*) Show the existence of a limiting distribution for an arbitrary graph G.

Question 3. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two probability distributions on a finite set X. Show that

$$d_{TV}(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}||\boldsymbol{p} - \boldsymbol{q}||_1.$$

Question 4. For any probability measure $\boldsymbol{p}$ show that $H_1(\boldsymbol{p}) \geq H_2(\boldsymbol{p}) \geq H_\infty(\boldsymbol{p})$.

Let X be a doubly stochastic matrix (non-negative entries with row and column sums equal to one). Show that $\tilde{H}\left(X\boldsymbol{p}\right) \geq \tilde{H}(\boldsymbol{p})$, with equality iff $\boldsymbol{p}$ is uniform.

Question 5. Let $G = (V, E)$ be an (n, d, α)-graph, $t \in \mathbb{N}$ and let $B_0, B_1, \ldots, B_t \subseteq V$ be subsets of cardinality $|B_i| = \beta_i n$. Let X_0 be chosen uniformly at random from $V(G)$ and let $X_0, X_1, \ldots$ be a random walk in G. Show that

$$\mathbb{P}(X_i \in B_i \text{ for all } i \in [t]) \leq \prod_{i=0}^{t-1} \left(\sqrt{\beta_i \beta_{i+1}} + \alpha\right)$$

and for any $K \subseteq \{0, 1 \ldots, t\}$

$$\mathbb{P}(X_i \in B_0 \text{ for all } i \in K) \leq (\beta_0 + \alpha)^{|K|-1}.$$